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Causal Identification under Markov Equivalence: Completeness Results

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Abstract

Causal effect identification is the task of determining whether a causal distribution is computable from the combination of an observational distribution and substantive knowledge about the domain under investigation. One of the most studied versions of this problem assumes that knowledge is articulated in the form of a fully known causal diagram, which is arguably a strong assumption in many settings. In this paper, we relax this requirement and consider that the knowledge is articulated in the form of an equivalence class of causal diagrams, in particular, a partial ancestral graph (PAG). This is attractive because a PAG can be learned directly from data, and the scientist does not need to commit to a particular, unique diagram. There are different sufficient conditions for identification in PAGs, but none is complete. We derive a complete algorithm for identification given a PAG. This implies that whenever the causal effect is identifiable, the algorithm returns a valid identification expression; alternatively, it will throw a failure condition, which means that the effect is provably not identifiable. We further provide a graphical characterization of non-identifiability of causal effects in PAGs.

1. Introduction

One cognitive feature that arguably distinguishes humans from other species is our ability to learn, process, and use causal information. Pearl recently highlighted this point eloquently: "Some tens of thousands of years ago, humans began to realize that certain things cause other things and that tinkering with the former can change the latter… From this discovery came organized societies, then towns and cities, and eventually the science and technology-based civilization we enjoy today. All because we asked a simple question: Why?" (Pearl & Mackenzie, 2018).

At the center of causal reasoning lies the idea of "tinkering" – what would happen if reality were different – which is formally materialized through cause and effect relations. Fisher proposed a procedure to physically manipulate reality such that an outcome variable can be evaluated under different conditions, which was called *randomized controlled experiments* (Fisher, 1951). This method is indeed one of the most pervasive techniques used throughout the sciences, and it is often deemed the gold standard for causal inference. For instance, the process of drug's approval by the FDA is conducted following Fisher's method – one can discover, for example, the effect of a drug (X) on survival (Y), which is written in causal language as the experimental distribution $P(y \mid do(x))$, or $P_x(y)$ for short.

Causal Identification. The infeasibility of always physically manipulating reality to see "what would happen" leads to one of the central challenges in causal inference, which is known as *the problem of identification of causal effects* (Pearl, 2000; Spirtes et al., 2001; Bareinboim & Pearl, 2016). It's well understood that no causal claim about $P_x(y)$ can be made directly from the observational distribution $P(\mathbf{V})$, where $\mathbf{V}$ is the set of measured (observed) variables (Pearl, 2000, Ch. 3). The idea is then to combine a coarse description of the underlying system, usually specified as a causal diagram ($\mathcal{D}$), with the observational data ($P(\mathbf{V})$) in order to infer the causal distribution $P_x(y)$.

A sample causal diagram is shown in Fig. 1a, where the nodes represent random variables, directed edges represent direct causal relations from tails to heads, and bi-directed arcs represent the presence of unobserved (latent) variables that generate spurious associations between the observed variables, also known as *confounding bias* (Pearl, 1993). One may seek to identify the effect of forcing variable X to take the value x, i.e., $do(X{=}x)$, on $V_4{=}v_4$, i.e., $P_x(v_4)$, given the causal diagram in Fig. 1a and data from the observational distribution $P(X, V_1, \ldots, V_4)$. In summary, the causal identifiability problem asks whether the combination of $\mathcal{D}$ and $P(\mathbf{V})$ allows the identification of $P_x(y)$; as in any

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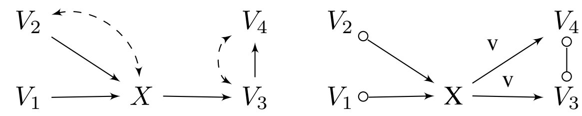
decision problem, the answer is sometimes negative, which happens whenever the causal diagram does not warrant a transformation of $P_x(y)$ into a functional of $P(\mathbf{V})$.

Identification has been extensively studied in the literature, and a number of criteria have been established (Pearl, 1993; Galles & Pearl, 1995; Kuroki & Miyakawa, 1999; Tian & Pearl, 2002; Huang & Valtorta, 2006; Shpitser & Pearl, 2006; Bareinboim & Pearl, 2012), including the celebrated back-door criterion and the do-calculus (Pearl, 1995). Despite their power, these techniques require a single, fully specified causal diagram, which is not always available in practical settings. A sensible concern, therefore, is that forcing a single diagram may lead to false modeling assumptions and, consequently, misleading inferences.

Markov Equivalence. Another line of investigation focuses precisely on trying to learn a qualitative description of the system, which in the ideal case could lead to the "true" causal diagram – the blueprint underlying the phenomenon being investigated. These efforts are usually deemed more "data-driven", and more aligned with the zeitgeist in machine learning. In practice, however, it is common that only a Markov equivalence class including a collection of causal diagrams can be consistently inferred from observational data (Verma, 1993; Spirtes et al., 2001; Zhang, 2008b). A distinguished characterization of the Markov equivalence class uses *partial ancestral graphs (PAGs)*. Fig. 1b shows the PAG learnable from observational data that is consistent with the true causal diagram (Fig. 1a). The directed edges in a PAG represent ancestral relations (not necessarily direct) and the circle marks stand for structural uncertainty.

Identification under Markov Equivalence. In this work, we analyze the marriage of these two lines of investigation, where the structural invariances in a Markov equivalence class (learnable from observational data) will be used to identify causal effects, whenever possible. However, identification from an equivalence class is considerably more challenging than from a single diagram due to the structural uncertainties. Given that fully specifying a causal diagram is usually infeasible, there is a growing interest in identifiability results under Markov Equivalence (Maathuis et al., 2010). For instance, Zhang (2007) extended the do-calculus to PAGs. In practice, however, it is computationally hard to decide whether there exists (and, if so, to find) a sequence of applications of the rules of the generalized calculus to identify the interventional distribution. Perković et al. (2015) generalized the back-door criterion to PAGs, and provided a sound and complete algorithm to find a back-door admissible set, should such a set exist. However, the back-door criterion is not as powerful as the do-calculus since no adjustment set exists for many identifiable causal effects. More recent work generalized the c-component approach (Tian & Pearl, 2002) to PAGs and devised an algorithm for identification under Markov equivalence (Jaber et al., 2018b;a). The algorithm was shown to be strictly more powerful than the generalized back-door criterion, but it is not complete, as we show later.[1]



(a) (b)

Figure 1: Causal diagram (left) and the inferred PAG (right).

Our strategy here is to combine the state-of-the-art graphical condition introduced in (Jaber et al., 2018a) with the classic c-component decomposition developed in (Tian, 2002) to obtain a PAG-specific decomposition, and use it to design a complete algorithm for identification in PAGs. It is worth noting that completeness is one of the most important guarantees a theory can offer – it provides a precise boundary between what is identifiable and what is not in the given setting. Specifically, our main contributions are as follows:

1. We introduce a novel graph-decomposition strategy that breaks the targeted causal distribution into an equivalent product of more amenable distributions.
2. We develop an algorithm to compute the effect of an arbitrary set of intervention variables on an arbitrary outcome set from a PAG and an observational distribution. We show that this algorithm is sound and complete.
3. We characterize non-identifiability in a PAG based on some forbidden graphical structures, which means that whenever such a structure can be found as a subgraph of the PAG, identification is provably impossible.

2. Preliminaries

In this section, we introduce the basic notation and machinery used throughout the paper. Bold capital letters denote sets of variables, while bold lowercase letters stand for particular assignments to those variables.

Structural Causal Models. We use the language of Structural Causal Models (SCMs) (Pearl, 2000, pp. 204-207) as our basic semantical framework. Formally, an SCM M is a 4-tuple $\langle \mathbf{U}, \mathbf{V}, \mathbf{F}, P(\mathbf{U})\rangle$, where $\mathbf{U}$ is a set of exogenous (latent) variables and $\mathbf{V}$ is a set of endogenous (measured) variables. $\mathbf{F}$ represents a collection of functions $\mathbf{F} = \{f_i\}$ such that each endogenous variable $V_i \in \mathbf{V}$ is determined by a function $f_i \in \mathbf{F}$, where f_i is a mapping from the re-

[1]Another approach is based on SAT (Boolean constraint satisfaction) solvers (Hyttinen et al., 2015). Given its somewhat distinct nature, a closer comparison lies outside the scope of this paper.

spective domain of $\mathbf{U}_i \cup \mathbf{Pa}_i$ to V_i, $\mathbf{U}_i \subseteq \mathbf{U}$, $\mathbf{Pa}_i \subseteq \mathbf{V} \setminus V_i$. The uncertainty is encoded through a probability distribution over the exogenous variables, $P(\mathbf{U})$, and the marginal distribution induced over the endogenous variables $P(\mathbf{V})$ is called observational. Every SCM is associated with one causal diagram where every variable $V_i \in \mathbf{V}$ is a node, and there exists a directed edge from every node in $\mathbf{Pa}_i$ to V_i. Also, for every pair $V_i, V_j \in \mathbf{V}$ such that $\mathbf{U}_i \cap \mathbf{U}_j \neq \emptyset$, there exists a bi-directed edge between V_i and V_j. We restrict our study to recursive systems, which means that the corresponding diagram will be acyclic.

Within the structural semantics, performing an action $X{=}x$ is represented through the do-operator, *do*$(X = x)$, which encodes the operation of replacing the original equation for X by the constant x and induces a submodel M_x. The resulting distribution is denoted by P_x, which is the main target for identification in this paper. For details on structural models, we refer readers to (Pearl, 2000).

Ancestral Graphs. We now introduce a graphical representation of equivalence classes of causal diagrams. A *mixed* graph can contain directed and bi-directed edges. A is an ancestor of B if they share a directed path out of A. A is a *spouse* of B if $A \leftrightarrow B$ is present. An *almost directed cycle* happens when A is both a spouse and an ancestor of B. An *inducing path* is a path on which every node (except for the endpoints) is a collider on the path (i.e., both edges incident to X are into X) and every collider is an ancestor of an endpoint of the path. A mixed graph is *ancestral* if it doesn't contain a directed or almost directed cycle. It is *maximal* if there is no inducing path (relative to the empty set) between any two non-adjacent nodes. A *Maximal Ancestral Graph* (MAG) is a graph that is both ancestral and maximal (Richardson & Spirtes, 2002).

In general, a causal MAG represents a set of causal diagrams with the same set of observed variables that entail the same independence and ancestral relations among the observed variables. Different MAGs may be Markov equivalent in that they entail the exact same independence model. A partial ancestral graph (PAG) represents an equivalence class of MAGs $[\mathcal{M}]$, which shares the same adjacencies as every MAG in $[\mathcal{M}]$ and displays all and only the invariant edge marks (i.e., edge marks that are shared by all members of $[\mathcal{M}]$). A circle indicates a edge mark that is not invariant.

A PAG is learnable from the conditional independence and dependence relations among the observed variables, and the FCI algorithm is a standard method to learn such an object (Zhang, 2008b). In short, a PAG represents a Markov equivalence class of causal diagrams with the same observed variables and independence model.

Graphical Notions. Given a causal diagram, MAG, or PAG, a path between X and Y is *potentially directed (causal)* from X to Y if there is no arrowhead on the path pointing towards X. Y is called a *possible descendant* of X and X a *possible ancestor* of Y, i.e., $X \in \texttt{An}(Y)$, if there is a potentially directed path from X to Y. By stipulation, $X \in \texttt{An}(X)$. A set $\mathbf{A}$ is (*descendant*) *ancestral* if no node outside $\mathbf{A}$ is a possible (descendant) ancestor of any node in $\mathbf{A}$. Y is called a *possible child* of X, i.e., $Y \in \texttt{Ch}(X)$, and X a *possible parent* of Y, i.e., $X \in \texttt{Pa}(Y)$, if they are adjacent and the edge is not into X. For a set of nodes $\mathbf{X}$, we have $\texttt{Pa}(\mathbf{X}) = \cup_{X \in \mathbf{X}} \texttt{Pa}(X)$ and $\texttt{Ch}(\mathbf{X}) = \cup_{X \in \mathbf{X}} \texttt{Ch}(X)$. Given two sets of nodes $\mathbf{X}$ and $\mathbf{Y}$, a path between them is called *proper* if one of the endpoints is in $\mathbf{X}$ and the other is in $\mathbf{Y}$, and no other node on the path is in $\mathbf{X}$ or $\mathbf{Y}$. For convenience, we use an asterisk (*) as a wildcard to denote any possible mark of a PAG ($\circ, >, -$) or a MAG ($>, -$). If the edge marks on a path between X and Y are all circles, we call the path a *circle path*. We refer to the closure of nodes connected with circle paths as a *bucket*. Obviously, given a PAG, nodes are partitioned into a unique set of buckets.

A directed edge $X \rightarrow Y$ in a MAG or PAG is *visible* if there exists no causal diagram in the corresponding equivalence class where there is an inducing path between X and Y that is into X. This implies that a visible edge is not confounded ($X \leftarrow\!-\!-\!\rightarrow Y$ doesn't exist). Which edges are visible is easily decidable by a graphical condition (Zhang, 2008a), so we simply mark visible edges by v. For brevity, we refer to any edge that is not a visible directed edge as *invisible*.

Identification in Causal Diagrams. Pearl (2000, pp. 70) formalizes the notion of uniquely computing an effect from data as follows.

Definition 1. *The causal effect of* $\mathbf{X}$ *on a disjoint set* $\mathbf{Y}$ *is said to be identifiable from a causal diagram* $\mathcal{D}$ *if* $P_{\mathbf{x}}(\mathbf{y})$ *can be computed uniquely from any positive probability of the observed variables* $P(\mathbf{V})$ *– that is, if* $P^{M_1}_{\mathbf{x}}(\mathbf{y}) = P^{M_2}_{\mathbf{x}}(\mathbf{y})$ *for every pair of models* M_1 *and* M_2 *with* $P^{M_1}(\mathbf{V}) = P^{M_2}(\mathbf{V}) > 0$ *and* $\mathcal{D}(M_1) = \mathcal{D}(M_2) = \mathcal{D}$.

Tian & Pearl (2002) presented an identification algorithm based on a decomposition of the causal diagram into a set of so-called *c-components* (confounded components).

Definition 2 (C-Component)**.** *In a causal diagram, two observed variables are said to be in the same c-component if and only if they are connected by a bi-directed path, i.e. a path composed solely of bi-directed edges.*

For any set $\mathbf{C} \subseteq \mathbf{V}$, the quantity $Q[\mathbf{C}]$ is defined to denote the post-intervention distribution of $\mathbf{C}$ under an intervention on $\mathbf{V} \setminus \mathbf{C}$, i.e. $P_{\mathbf{v} \setminus \mathbf{c}}(\mathbf{c})$.

The significance of c-components and their decomposition is evident from (Tian, 2002, Lemmas 10, 11), which are the basis of the complete algorithm relative to Definition 1.

Identification in PAGs. The following definition formalizes the notion of identification given a PAG, which generalizes the diagram-specific notion in Definition 1.

Definition 3. *Given a PAG $\mathcal{P}$ over $\mathbf{V}$ and a query $P_{\mathbf{x}}(\mathbf{y})$ where $\mathbf{X}, \mathbf{Y} \subset \mathbf{V}$, $P_{\mathbf{x}}(\mathbf{y})$ is identifiable given $\mathcal{P}$ if and only if $P_{\mathbf{x}}(\mathbf{y})$ is identifiable given every causal diagram $\mathcal{D}$ (represented by a MAG) in the Markov equivalence class represented by $\mathcal{P}$, and with the same expression.*

Jaber et al. (2018a) introduced the notion of pc-component in MAGs and PAGs (and induced subgraphs thereof). This graphical construction is a necessary condition for two nodes to be in the same c-component in some causal diagram or an induced subgraph thereof. This observation is formalized in the proposition below.

Definition 4 (PC-Component)**.** *In a MAG, a PAG, or any induced subgraph thereof, two nodes are in the same possible c-component (pc-component) if there is a path between them such that (1) all non-endpoint nodes along the path are colliders, and (2) none of the edges is visible.*

Proposition 1. *Let $\mathcal{P}$ be a PAG over $\mathbf{V}$, and $\mathcal{D}$ be any diagram in the equivalence class represented by $\mathcal{P}$. For any $X, Y \in \mathbf{A} \subseteq \mathbf{V}$, if X and Y are in the same c-component in $\mathcal{D}_{\mathbf{A}}$, then X and Y are in the same pc-component in $\mathcal{P}_{\mathbf{A}}$.*

As a special case of Def. 4, we have the following notion, which will prove useful later on. Unlike *pc-component*, the relation of being in the same *dc-component* is transitive.

Definition 5 (DC-Component)**.** *In a MAG, a PAG, or any induced subgraph thereof, two nodes are in the same definite c-component (dc-component) if they are connected with a bi-directed path, i.e. a path of bi-directed edges.*

Jaber et al. (2018a) developed an identification criterion for PAGs where the intervention is on a bucket and the input distribution is possibly interventional. We introduce this result below as it will be used in our algorithm. The derived expression depends on a partial topological order (*PTO*) which is, in short, a topological order over the buckets. A detailed discussion can be found in (Jaber et al., 2018b).

Proposition 2. *Let $\mathcal{P}$ denote a PAG over $\mathbf{V}$, $\mathbf{T}$ be the union of a subset of the buckets in $\mathcal{P}$, and $\mathbf{X} \subset \mathbf{T}$ be a bucket. Given $P_{\mathbf{v}\setminus\mathbf{t}}$ (i.e., $Q[\mathbf{T}]$), and a partial topological order $\mathbf{B}_1 < \cdots < \mathbf{B}_m$ with respect to $\mathcal{P}_{\mathbf{T}}$, $Q[\mathbf{T}\setminus\mathbf{X}]$ is identifiable if and only if, in $\mathcal{P}_{\mathbf{T}}$, there does not exist $Z \in \mathbf{X}$ such that Z has a possible child $C \notin \mathbf{X}$ that is in the pc-component of Z. If identifiable, then the expression is given by*

$$Q[\mathbf{T}\setminus\mathbf{X}] = \frac{P_{\mathbf{v}\setminus\mathbf{t}}}{\prod_{\{i|\mathbf{B_i}\subseteq S^{\mathbf{X}}\}} P_{\mathbf{v}\setminus\mathbf{t}}(\mathbf{B_i}|\mathbf{B^{(i-1)}})} \times \sum_{\mathbf{x}} \prod_{\{i|\mathbf{B_i}\subseteq S^{\mathbf{X}}\}} P_{\mathbf{v}\setminus\mathbf{t}}(\mathbf{B_i}|\mathbf{B^{(i-1)}}), \tag{1}$$

where $S^{\mathbf{X}} = \bigcup_{Z\in\mathbf{X}} S^Z$, S^Z being the dc-component of Z in $\mathcal{P}_{\mathbf{T}}$, and $\mathbf{B}^{(i-1)}$ denoting the set of nodes preceding bucket $\mathbf{B}_i$ in the partial order.

3. Query Decomposition

Given a causal diagram $\mathcal{D}$, one of the cornerstone results in (Tian, 2002) is Lemma 11, which allows one to decompose a query distribution $Q[\mathbf{H}]$ into a product of sub-queries over the c-components in $\mathcal{D}_{\mathbf{H}}$. Hence, we get the following decomposition, where $\mathbf{H}_i$ is a c-component in $\mathcal{D}_{\mathbf{H}}$:

$$Q[\mathbf{H}] = \prod_i Q[\mathbf{H}_i] \tag{2}$$

For example, query $Q[\{Y_1, Y_2, Y_3, Y_4, Y_5\}]$, denoted $Q[\mathbf{Y}]$, given the causal diagram in Fig. 2b can be decomposed as follows:

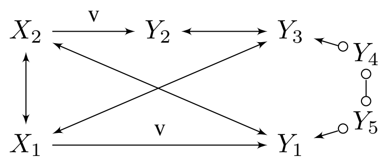
$$Q[\mathbf{Y}] = Q[\{Y_2, Y_3\}] \,.\, Q[\{Y_4, Y_5\}] \,.\, Q[\{Y_1\}] \tag{3}$$

We note that the decomposition relies heavily on the precise delimitation of the c-components, where each query is associated, respectively, with $H_1 = \{Y_2, Y_3\}$, $H_2 = \{Y_4, Y_5\}$, and $H_3 = \{Y_1\}$. Generalizing the decomposition to PAGs, therefore, is challenging given the structural uncertainties; most relevant the presence of latent confounders. A decomposition has to be valid in every causal diagram in the equivalence class. For instance, given the query $Q[\mathbf{Y}]$ over the PAG in Fig. 2a, the sequence of nodes $\langle Y_2, Y_3, Y_4, Y_5, Y_1\rangle$ is connected with invisible edges, which are possibly confounded. Hence, any naive decomposition of $Q[\mathbf{Y}]$ into a product of sub-queries over subsets of $\mathbf{Y}$ is invalid since we can construct a diagram in the equivalence class which violates this decomposition. For instance, the decomposition in Eq. 3 is valid for the diagram in Fig. 2b but not for the one in Fig. 2c. Yet, we can still decompose the query using some invariances such as the fact that Y_2 and Y_1 are not in the same pc-component in $\mathcal{P}_{\mathbf{Y}}$, hence they are not in the same c-component in the $\mathbf{Y}$-induced subgraph of any causal diagram in the equivalence class by Prop. 1.

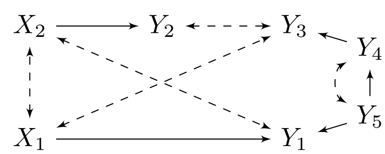
To develop an invariantly valid decomposition, we start by introducing the notion of a *region*. In short, a *region* is the pc-component of a set $\mathbf{A}$ appended with the corresponding buckets of the nodes. The pc-component of a set $\mathbf{A}$ includes all the nodes which could, in some causal diagram, be in the c-component of some node in $\mathbf{A}$ (Prop. 1). We append the pc-component set with the corresponding buckets of the nodes to avoid non-identifiability of the sub-queries since no sufficient causal information is present within a bucket.

Definition 6 (Region $\mathcal{R}^{\mathbf{C}}_{\mathbf{A}}$)**.** *Given a PAG or a MAG $\mathcal{G}$ over $\mathbf{V}$, and $\mathbf{A} \subseteq \mathbf{C} \subseteq \mathbf{V}$. Let the region of $\mathbf{A}$ with respect to $\mathbf{C}$, denoted $\mathcal{R}^{\mathbf{C}}_{\mathbf{A}}$, be the union of the buckets that contain nodes in the pc-component of $\mathbf{A}$ in the induced subgraph $\mathcal{G}_{\mathbf{C}}$.*

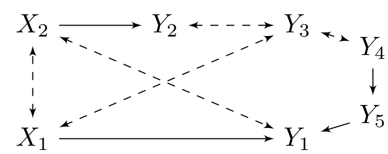
Consider the PAG in Figure 2a, and let $\mathbf{C} = \mathbf{Y}$ and $\mathbf{A} = \{Y_3\}$. Then, $\mathcal{R}^{\mathbf{C}}_{\mathbf{A}} = \{Y_3, Y_2, Y_4, Y_5\}$ since Y_2 and Y_4 are in the pc-component of Y_3 and Y_5 is in the same bucket as Y_4. Alternately, if $\mathbf{A} = \{Y_4, Y_5\}$, then $\mathcal{R}^{\mathbf{C}}_{\mathbf{A}} = \mathbf{Y}$,



(a) PAG $\mathcal{P}$.



(b) 1st causal diagram.



(c) 2nd causal diagram.

Figure 2: Sample PAG with two causal diagrams in the equivalence class.

since Y_2 and Y_3 are in the pc-component of Y_4 and Y_1 is in the pc-component of Y_5. For simplicity, we often drop $\mathbf{C}$, i.e. $\mathcal{R}_{\mathbf{A}}$, whenever it is clear from the context. Using this construction, we derive a useful decomposition as follows.

Theorem 1. *Given a PAG $\mathcal{P}$ over $\mathbf{V}$ and set $\mathbf{C} \subseteq \mathbf{V}$, $Q[\mathbf{C}]$ can be decomposed as follows.*

$$Q[\mathbf{C}] = \frac{Q[\mathbf{A}]\ .\ Q[\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}]}{Q[\mathbf{A} \cap \mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}]} \tag{4}$$

where $\mathbf{A} \subset \mathbf{C}$ and $\mathcal{R}_{(.)} = \mathcal{R}^{\mathbf{C}}_{(.)}$.

Proof. Let $\mathcal{D}$ be any causal diagram in the equivalence class of $\mathcal{P}$. We show that Eq. 4 is valid for $\mathcal{D}$. Let $\mathcal{D}_{\mathbf{C}}$ be the $\mathbf{C}$-induced subgraph of $\mathcal{D}$. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ partition $\mathbf{A} \cap \mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}$ in $\mathcal{D}_{\mathbf{C}}$ where $\mathbf{S}_1$ contains nodes that are in the c-component of some node in $\mathbf{A} \setminus \mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}$, and $\mathbf{S}_2$ contains the rest.

By construction of $\mathbf{S}_1$ and $\mathbf{S}_2$, no node in $\mathbf{S}_1$ is in the same c-component as any node in $\mathbf{S}_2$ in $\mathcal{D}_{\mathbf{A}\cap\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}}$, so $Q[\mathbf{A} \cap \mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}] = Q[\mathbf{S}_1] \times Q[\mathbf{S}_2]$. Moreover, $Q[\mathbf{A}] = Q[\mathbf{A} \setminus \mathbf{S}_2] \times Q[\mathbf{S}_2]$ since none of the nodes in $\mathbf{S}_2$ is in the c-component of $\mathbf{A} \setminus \mathbf{S}_2$ in $\mathcal{D}_{\mathbf{C}}$, and consequently in $\mathcal{D}_{\mathbf{A}}$ since $\mathbf{A} \subset \mathbf{C}$. Also, we claim that $Q[\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}] = Q[\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}} \setminus \mathbf{S}_1] \times Q[\mathbf{S}_1]$. Suppose for contradiction that the claim is not true, then some node $S \in \mathbf{S}_1$ is in the c-component of $X \in \mathcal{R}_{\mathbf{C}\setminus\mathbf{A}} \setminus \mathbf{S}_1 = (\mathbf{C} \setminus \mathbf{A}) \cup \mathbf{S}_2$ in $\mathcal{D}_{\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}}$. But, $X \notin \mathbf{S}_2$ since $\mathbf{S}_1$ and $\mathbf{S}_2$ are in different c-component in $\mathcal{D}_{\mathbf{C}}$, and consequently in $\mathcal{D}_{\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}}$. So $X \in \mathbf{C} \setminus \mathbf{A}$. But then we have the following in $\mathcal{D}_{\mathbf{C}}$: X is in the c-component of S, and S is in the c-component of some node $Y \in \mathbf{A} \setminus \mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}$ by definition of $S \in \mathbf{S}_1$. This is a contradiction since Y would then be in the pc-component of X in the induced sub-PAG $\mathcal{P}_{\mathbf{C}}$ (Prop. 1), and consequently part of $\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}$. Therefore, the claimed decomposition of $Q[\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}]$ is also valid in $\mathcal{D}$. By the previous observations in $\mathcal{D}$, we can simplify the right-hand side of Eq. 4 as follows:

$$\begin{aligned}\frac{Q[\mathbf{A}].Q[\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}]}{Q[\mathbf{A} \cap \mathcal{R}_{\mathbf{C}\setminus\mathbf{A}}]} &= \frac{Q[\mathbf{A} \setminus \mathbf{S}_2].Q[\mathbf{S}_2].Q[\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}} \setminus \mathbf{S}_1].Q[\mathbf{S}_1]}{Q[\mathbf{S}_1].Q[\mathbf{S}_2]} \\ &= Q[\mathbf{A} \setminus \mathbf{S}_2].Q[\mathcal{R}_{\mathbf{C}\setminus\mathbf{A}} \setminus \mathbf{S}_1]\end{aligned} \tag{5}$$

Note that Eq. 5 is equivalent to $Q[\mathbf{A} \setminus \mathbf{S}_2].Q[(\mathbf{C} \setminus \mathbf{A}) \cup \mathbf{S}_2]$. By the previous derivation, it is easy to verify that no node in $\mathbf{A} \setminus \mathbf{S}_2$ is in the same c-component as any node in $(\mathbf{C} \setminus \mathbf{A}) \cup \mathbf{S}_2$. Hence, the decomposition in Eq. 4 is valid for $\mathcal{D}$ as the right-hand side can be simplified to be consistent with Eq. 2. This concludes the proof. □

Back to the query $Q[\mathbf{Y}]$ over the PAG in Fig. 2a, we can decompose the query accordingly with $\mathbf{A} = \{Y_2, Y_3, Y_4, Y_5\}$:

$$Q[\mathbf{Y}] = \frac{Q[\mathbf{A}].Q[\mathcal{R}^{\mathbf{Y}}_{\mathbf{Y}\setminus\mathbf{A}}]}{Q[\mathbf{A} \cap \mathcal{R}^{\mathbf{Y}}_{\mathbf{Y}\setminus\mathbf{A}}]} = \frac{Q[\mathbf{Y} \setminus \{Y_1\}].Q[\{Y_1, Y_4, Y_5\}]}{Q[\{Y_4, Y_5\}]} \tag{6}$$

Alternately, we get this decomposition for $\mathbf{A} = \{Y_1\}$:

$$Q[\mathbf{Y}] = \frac{Q[\{Y_1\}]\ .\ Q[\{Y_1, Y_2, Y_3, Y_4, Y_5\}]}{Q[\{Y_1\}]} = Q[\mathbf{Y}] \tag{7}$$

The above two examples show that the decomposition is sensitive to the choice of $\mathbf{A}$, and consequently raises a couple of interesting issues about Theorem 1:

1. Some decompositions are useless, e.g. Eq. 7.

2. Some queries in the decomposition are not identifiable solely due to the choice of $\mathbf{A}$, e.g. $Q[\{Y_1\}]$ in Eq. 7 will turn out to be not identifiable due to the invisible edge $Y_5 \circ\!\!\rightarrow Y_1$.

The following decomposition is a special case of Thm. 1 and allows us to overcome these weaknesses. Note that Eq. 6 follows from this corollary with $\mathbf{A} = \{Y_3\}$.

Corollary 1. *Given a PAG $\mathcal{P}$ over $\mathbf{V}$ and set $\mathbf{C} \subseteq \mathbf{V}$, $Q[\mathbf{C}]$ can be decomposed as follows.*

$$Q[\mathbf{C}] = \frac{Q[\mathcal{R}_{\mathbf{A}}].Q[\mathcal{R}_{\mathbf{C}\setminus\mathcal{R}_{\mathbf{A}}}]}{Q[\mathcal{R}_{\mathbf{A}} \cap \mathcal{R}_{\mathbf{C}\setminus\mathcal{R}_{\mathbf{A}}}]}$$

where $\mathbf{A} \subset \mathbf{C}$ and $\mathcal{R}_{(.)} = \mathcal{R}^{\mathbf{C}}_{(.)}$.

Proof. It follows from Thm. 1 with $\mathbf{A}$ replaced by $\mathcal{R}^{\mathbf{C}}_{\mathbf{A}}$. □

4. A Complete Identification Algorithm

Using the identification criterion in Prop. 2 and the decomposition in Corol. 1, we formulate the procedure we call **IDP**, which is shown in Alg. 1. The main idea of **IDP** goes as follows. After receiving the sets $\mathbf{X}$, $\mathbf{Y}$, and a PAG $\mathcal{P}$, the algorithm pre-processes the query by computing $\mathbf{D}$, the set of possible ancestors of $\mathbf{Y}$ in $\mathcal{P}_{\mathbf{V}\setminus\mathbf{X}}$. Then, the procedure calls the subroutine `Identify` over $\mathbf{D}$ to compute $Q[\mathbf{D}]$ from the observational distribution $P(\mathbf{V})$. The recursive routine basically tests for one of two conditions. First, it checks for the presence of a bucket $\mathbf{B}$ in $\mathcal{P}_{\mathbf{T}}$ that is a subset of the intervention nodes, i.e. $\mathbf{B} \subseteq \mathbf{T} \setminus \mathbf{C}$, and satisfies the conditions of Prop. 2. If found, it computes $Q[\mathbf{T} \setminus \mathbf{B}]$ using Eq. 1, and proceeds with a recursive call. Alternatively, if such a bucket does not exist in $\mathcal{P}_{\mathbf{T}}$, then **IDP** checks for a bucket $\mathbf{B}$ in $\mathcal{P}_{\mathbf{C}}$ such that the region of $\mathbf{B}$ with respect to $\mathbf{C}$, i.e. $\mathcal{R}^{\mathbf{C}}_{\mathbf{B}}$, does not span $\mathbf{C}$. If such a bucket exists, then **IDP** decomposes the query $Q[\mathbf{C}]$ according to Corol. 1. Finally, if both tests fail, then **IDP** throws a failure condition.

Theorem 2. **IDP** *(Alg. 1) terminates and is sound.*

Proof. Starting with termination, every recursive call of `Identify` strictly decreases the size of the input sets $\mathbf{C}$ and $\mathbf{T}$. This is evident in the call at line (8). We are left with the three recursive calls at line (10) due to the decomposition (Corol. 1). $\mathcal{R}_{\mathbf{B}}$ is a strict subset of $\mathbf{C}$ by construction. Then, $\mathcal{R}_{\mathbf{C}\setminus\mathcal{R}_{\mathbf{B}}}$ is a strict subset of $\mathbf{C}$ as well since the pc-component property is symmetric. Finally, it follows easily that $\mathcal{R}_{\mathbf{B}} \cap \mathcal{R}_{\mathbf{C}\setminus\mathcal{R}_{\mathbf{B}}}$ is a strict subset of $\mathbf{C}$.

Given that **IDP** terminates, we now move to soundness. Let $\mathcal{G}$ be any causal diagram in the equivalence class of PAG $\mathcal{P}$ over $\mathbf{V}$, and let $\mathbf{V}' = \mathbf{V} \setminus \mathbf{X}$. We have

$$P_{\mathbf{x}}(\mathbf{y}) = \sum_{\mathbf{v}'\setminus\mathbf{y}} P_{\mathbf{x}}(\mathbf{v}') = \sum_{\mathbf{v}'\setminus\mathbf{y}} Q[\mathbf{V}'] = \sum_{\mathbf{v}'\setminus\mathbf{d}} \sum_{\mathbf{d}\setminus\mathbf{y}} Q[\mathbf{V}']$$

By definition, $\mathbf{D}$ is an ancestral set in $\mathcal{P}_{\mathbf{V}'}$, and hence it is ancestral in $\mathcal{G}_{\mathbf{V}'}$ by (Jaber et al., 2018a, Prop. 1). So, we have the following by (Tian, 2002, Lem. 10):

$$P_{\mathbf{x}}(\mathbf{y}) = \sum_{\mathbf{d}\setminus\mathbf{y}} \sum_{\mathbf{v}'\setminus\mathbf{d}} Q[\mathbf{V}'] = \sum_{\mathbf{d}\setminus\mathbf{y}} Q[\mathbf{D}] \tag{8}$$

Eq. 8 is equivalent to the expression in line (2) of Alg. 1. Finally, the correctness of the recursive routine `Identify` follows from that of Proposition 2 and Corollary 1. □

4.1. Illustrative example

Consider the query $P_{\mathbf{x}}(\mathbf{y})$ given PAG $\mathcal{P}$ in Fig. 2a, where $\mathbf{X} = \{X_1, X_2\}$ and $\mathbf{Y} = \{Y_1, Y_2, Y_3, Y_4, Y_5\}$. We have $\mathbf{D} = \mathbf{Y}$, and the problem reduces to computing $Q[\mathbf{D}]$ using

Algorithm 1 **IDP**$(\mathbf{x}, \mathbf{y})$ given PAG $\mathcal{P}$

input two disjoint sets $\mathbf{X}, \mathbf{Y} \subset \mathbf{V}$
output Expression for $P_{\mathbf{x}}(\mathbf{y})$ or `FAIL`

1: Let $\mathbf{D} = \texttt{An}(\mathbf{Y})_{\mathcal{P}_{\mathbf{V}\setminus\mathbf{X}}}$
2: $P_{\mathbf{x}}(\mathbf{y}) = \sum_{\mathbf{d}\setminus\mathbf{y}} \texttt{Identify}(\mathbf{D}, \mathbf{V}, P)$
3: **function** `Identify` $(\mathbf{C}, \mathbf{T}, Q = Q[\mathbf{T}])$
4: **if** $\mathbf{C} = \emptyset$ **then return** 1
5: **if** $\mathbf{C} = \mathbf{T}$ **then return** Q
/ In $\mathcal{P}_{\mathbf{T}}$, let $\mathbf{B}$ denote a bucket, and let $C^{\mathbf{B}}$ denote the pc-component of $\mathbf{B}$ */*
6: **if** $\exists \mathbf{B} \subset \mathbf{T} \setminus \mathbf{C}$ such that $C^{\mathbf{B}} \cap \texttt{Ch}(\mathbf{B}) \subseteq \mathbf{B}$ **then**
7: Compute $Q[\mathbf{T} \setminus \mathbf{B}]$ from Q (via Prop. 2)
8: **return** $\texttt{Identify}(\mathbf{C}, \mathbf{T} \setminus \mathbf{B}, Q[\mathbf{T} \setminus \mathbf{B}])$
9: **else if** $\exists \mathbf{B} \subset \mathbf{C}$ such that $\mathcal{R}_{\mathbf{B}} \neq \mathbf{C}$ **then**
10: **return** $\frac{\texttt{Identify}(\mathcal{R}_{\mathbf{B}}, \mathbf{T}, Q)\,\cdot\,\texttt{Identify}(\mathcal{R}_{\mathbf{C}\setminus\mathcal{R}_{\mathbf{B}}}, \mathbf{T}, Q)}{\texttt{Identify}(\mathcal{R}_{\mathbf{B}}\cap\mathcal{R}_{\mathbf{C}\setminus\mathcal{R}_{\mathbf{B}}}, \mathbf{T}, Q)}$
11: **else**
12: **throw** `FAIL`
13: **end if**
14: **end function**

the call $\texttt{Identify}(\mathbf{D}, \mathbf{V}, P)$. None of the buckets X_1 and X_2 satisfy the condition at line (6) of Alg. 1, hence **IDP** tries to decompose the query. Bucket Y_3 satisfies the condition at line (9) as $\mathcal{R}^{\mathbf{Y}}_{Y_3} = \{Y_2, Y_3, Y_4, Y_5\} \subset \mathbf{Y}$, and we have the decomposition derived earlier in Eq. 6.

First, with the call $\texttt{Identify}(\{Y_2, Y_3, Y_4, Y_5\}, \mathbf{V}, P)$, node Y_1 satisfies the test at line (6) as it has no children. Hence, we compute $Q[\mathbf{V} \setminus \{Y_1\}]$ using Proposition 2.

$$\begin{aligned} Q[\mathbf{V} \setminus \{Y_1\}] &= \frac{P(\mathbf{v})}{P(y_1, y_2, y_3, x_1, x_2 | y_4, y_5)} \times \\ &\qquad \sum_{y_1} P(y_1, y_2, y_3, x_1, x_2 | y_4, y_5) \\ &= P(y_2, y_3, y_4, y_5, x_1, x_2) \end{aligned} \tag{9}$$

In the next recursive call, let $\mathbf{T}_1 = \mathbf{V} \setminus \{Y_1\}$ and P_{y_1} corresponds to Eq. 9. Now, X_1 satisfies the condition at line (6), and we can compute $Q[\mathbf{T}_1 \setminus \{X_1\}]$ from $P_{y_1} = Q[\mathbf{T}_1]$,

$$\begin{aligned} Q[\mathbf{T}_1 \setminus \{X_1\}] &= \frac{P_{y_1}}{P_{y_1}(y_2, y_3, x_1, x_2 | y_4, y_5)} \times \\ &\qquad \sum_{x_1} P_{y_1}(y_2, y_3, x_1, x_2 | y_4, y_5) \\ &= P(y_2, y_3, y_4, y_5, x_2) \end{aligned} \tag{10}$$

Let $\mathbf{T}_2 = \mathbf{T}_1 \setminus \{X_1\}$. Now, X_2 satisfies the identification criterion and we can compute $Q[\mathbf{T}_2 \setminus \{X_2\}]$ from Eq. 10,

$$Q[\mathbf{T}_2 \setminus \{X_2\}] = \frac{P_{x_1,y_1}}{P_{x_1,y_1}(x_2)} \times \sum_{x_2} P_{x_1,y_1}(x_2)$$
$$= P(y_2, y_3, y_4, y_5|x_2) \quad (11)$$

Similarly, we get the following expressions for $Q[\{Y_1, Y_4, Y_5\}]$ and $Q[\{Y_4, Y_5\}]$, respectively.

$$Q[\{Y_1, Y_4, Y_5\}] = P(y_1, y_4, y_5|x_1) \quad (12)$$

and,

$$Q[\{Y_4, Y_5\}] = P(y_4, y_5) \quad (13)$$

Hence, the final expression for $P_{\mathbf{x}}(\mathbf{y})$ is a result of Eqs. 11, 12, and 13 as follows.

$$P_{\mathbf{x}}(\mathbf{y}) = \frac{P(y_2, y_3, y_4, y_5|x_2) \ . \ P(y_1, y_4, y_5|x_1)}{P(y_4, y_5)} \quad (14)$$

We can simplify Eq. 14 using independence relations to,

$$P_{\mathbf{x}}(\mathbf{y}) = P(y_3|x_2, y_2, y_4)P(y_2|x_2)P(y_1|x_1, y_5)P(y_4, y_5)$$

Note that the adjustment criterion in (Perković et al., 2015) and the algorithm in (Jaber et al., 2018a) fail to compute the above causal effect, and hence are not complete.

4.2. Completeness

After introducing the identification algorithm and proving its soundness, we turn to the completeness of the procedure. According to Def. 3, whenever **IDP** fails, we need to establish one of two conditions for completeness. Either there exist two causal diagrams in the equivalence class with different identifications, or the effect is not identifiable in some causal diagram, where a hedge for $P_{\mathbf{x}}(\mathbf{y})$ exists (Shpitser & Pearl, 2006, Def. 7). In this section, we establish completeness by proving that the latter is always the case.

The blueprint for the proof goes as follows. First, we show that the recursive routine `Identify` fails only if an intervention node from $\mathbf{X}$ remains in $\mathbf{T}$. Then, we construct a MAG in the equivalence class of PAG $\mathcal{P}$, and we prove that an induced subgraph of it has some special properties (to be defined). Finally, we use the latter MAG to construct a causal diagram in the corresponding equivalence class with a hedge for $P_{\mathbf{x}}(\mathbf{y})$. This concludes the proof.

Lemma 1. *Whenever **IDP** fails, there exist at least one node $X \in \mathbf{X}$ such that $X \in \mathbf{T}$ in the failing instance of* `Identify`$(\mathbf{C}, \mathbf{T}, Q)$.[2]

[2]See (Jaber et al., 2019) for all the proofs.

We first consider an easy case of non-identifiable causal effects in the following theorem.

Theorem 3. *Given PAG $\mathcal{P}$ over $\mathbf{V}$ and query $P_{\mathbf{x}}(\mathbf{y})$ where $\mathbf{X}, \mathbf{Y} \subset \mathbf{V}$, if there exist a proper possibly directed path from $\mathbf{X}$ to $\mathbf{Y}$ that starts with an invisible edge (i.e. $*\!\!\rightarrow$, $\circ\!\!-\!\!\circ$), then $P_{\mathbf{x}}(\mathbf{y})$ is not identifiable.*

Proof. By (Perković et al., 2015, Lem. 5.5), we construct MAG $\mathcal{M}$ in the equivalence class of $\mathcal{P}$ with a proper directed path from $\mathbf{X}$ to $\mathbf{Y}$ that starts with invisible $X \rightarrow R$. Then, we construct a causal diagram in the equivalence class of $\mathcal{M}$ retaining the directed path and where $X \rightarrow R$ is confounded (Zhang, 2008a, Lem. 10). $\mathbf{F} = \{X, R\}, \mathbf{F}' = \{R\}$ form a hedge for $P_{\mathbf{x}}(\mathbf{y})$ (Shpitser & Pearl, 2006, Th. 4). ☐

In what follows, we assume the following condition holds, otherwise the case is covered by Thm. 3.

Condition 1. *Assume $\mathbf{IDP}(\mathbf{x}, \mathbf{y})$ fails, then every proper possibly directed path from $\mathbf{X}$ to $\mathbf{Y}$ in PAG $\mathcal{P}$ starts with a visible edge.*

This implies that every bucket in PAG $\mathcal{P}$ is a subset of set $\mathbf{D}$ in line (1) of **IDP** or $\mathbf{V} \setminus \mathbf{D}$. Since Prop. 2 and Corol. 1 don't split any bucket, then every bucket in $\mathcal{P}_{\mathbf{T}}$, local to the failing call of `Identify`$(\mathbf{C}, \mathbf{T}, Q)$, is a subset of $\mathbf{C}$ or $\mathbf{T} \setminus \mathbf{C}$. With these observations, we construct a MAG $\mathcal{M}$ in the equivalence class of the input PAG $\mathcal{P}$ using the procedure in Lemma 2.

Lemma 2. *Under the failing conditions of $\mathbf{IDP}(\mathbf{x}, \mathbf{y})$, i.e. a recursive* `Identify`$(\mathbf{C}, \mathbf{T}, Q)$ *fails, we can construct a MAG $\mathcal{M}$ in the equivalence class of PAG $\mathcal{P}$ resulting from the following procedure applied to $\mathcal{P}$:*

1. *orient the circles on $\circ\!\!\rightarrow$ edges in $\mathcal{P}$ as tails;*
2. *For bucket $\mathbf{B} \subset \mathcal{P}_{\mathbf{V} \setminus \mathbf{T}}$, orient into a DAG with no unshielded colliders;*
3. *For bucket $\mathbf{B} \subset \mathcal{P}_{\mathbf{T} \setminus \mathbf{C}}$, orient all edges out of some node $B \in \mathbf{B}$ such that B is in the same pc-component with a possible child $W \notin \mathbf{B}$ in $\mathcal{P}_{\mathbf{T}}$; and*
4. *For buckets in $\mathcal{P}_{\mathbf{C}}$, let $\mathbf{B}_1 < \cdots < \mathbf{B}_m$ be PTO over $\mathcal{P}_{\mathbf{C}}$:*
   (a) In $\mathbf{B}_m$, orient all edges out of any node $B^* \in \mathbf{B}_m$.
   (b) For every $\mathbf{B}_i$, $1 \leq i < m$, orient all edges out of some node $B \in \mathbf{B}_i$ such that B is in the same pc-component with B^* in $\mathcal{P}_{\mathbf{C}}$.

In the construction of Lem. 2, the intent behind the choice of the nodes and the edge orientations is to maintain the failing conditions of the induced subgraph $\mathcal{P}_{\mathbf{T}}$ in an induced subgraph of $\mathcal{M}$. Those properties are established in Lem. 3.

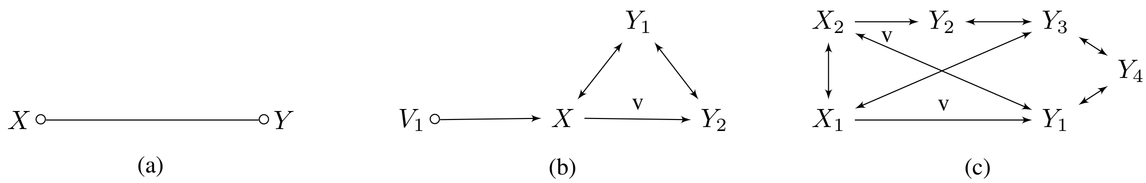


Figure 3: Sample PAGs where $P_{\mathbf{x}}(\mathbf{y})$ is not identifiable.

Lemma 3. *Let $\mathcal{M}$ denote the MAG constructed in Lem. 2. Then, there is a corresponding induced subgraph $\mathcal{M}_{\mathbf{T}'}$, with $\mathbf{C}' \subset \mathbf{T}' \subset \mathbf{V}$, that maintains the following properties:*

1. *$\mathbf{C}' \subseteq An(\mathbf{Y})_{\mathcal{M}_{\mathbf{V}\setminus\mathbf{X}}}$;*
2. *$\mathbf{C}'$ is a single dc-component in $\mathcal{M}_{\mathbf{C}'}$;*
3. *$\mathbf{T}' \setminus \mathbf{C}'$ contains at least one intervention $\mathcal{X} \in \mathbf{X}$; and*
4. *every node in $\mathbf{T}' \setminus \mathbf{C}'$ is in the dc-component with a child.*

With the previous observations under Condition 1, we establish the following non-identifiability result which is complementary to Th. 3.

Theorem 4. *Assume* **IDP** *fails to identify $P_{\mathbf{x}}(\mathbf{y})$ in PAG $\mathcal{P}$, then there exist a causal diagram in the equivalence class of $\mathcal{P}$ such that $P_{\mathbf{x}}(\mathbf{y})$ is not identifiable.*

Proof. Lem. 2 constructs a MAG $\mathcal{M}$ in the equivalence class of $\mathcal{P}$. According to Lem. 3, an induced subgraph of $\mathcal{M}$ over $\mathbf{T}'$ maintains the given properties. We construct a causal diagram $\mathcal{D}$ from $\mathcal{M}$ by simply keeping all directed and bi-directed edges intact. The diagram is trivially in the equivalence class of $\mathcal{M}$. In $\mathcal{D}_{\mathbf{T}'}$, $\mathcal{D}_{\mathbf{C}'}$ is a single c-component, and every node in $\mathbf{T}' \setminus \mathbf{C}'$ is in the c-component with a child. It follows that $\mathbf{T}' = \texttt{An}(\mathbf{C}')$ and $\mathbf{T}'$ is a single c-component, otherwise some node in $\mathbf{T}' \setminus \mathbf{C}'$ is not in the c-component with a child. Let $\mathbf{R}$ be the root set of $\mathbf{C}'$, then $\mathbf{R}$ is a root set of $\mathbf{T}'$ as well. We can remove directed edges from $\mathcal{D}_{\mathbf{T}'}$ so that $\mathbf{C}'$ and $\mathbf{T}'$ form $\mathbf{R}$-rooted C-forests (Shpitser & Pearl, 2006, Def. 6). Finally, $\mathbf{T}' \setminus \mathbf{C}'$ contains at least one intervention $\mathcal{X} \in \mathbf{X}$ and $\mathbf{R} \subseteq \mathbf{C}' \subseteq \texttt{An}(\mathbf{Y})_{\mathcal{D}_{\mathbf{V}\setminus\mathbf{X}}}$. Hence, $\mathbf{T}'$, $\mathbf{C}'$ construct a hedge for $P_{\mathbf{x}}(\mathbf{y})$ in $\mathcal{D}$ and the effect is not identifiable. □

Corollary 2. **IDP** *(Alg. 1) is complete.*

Proof. This follows from Theorems 3 and 4. □

5. Non-Identifiable Causal Effects

In this section, we aim to derive a graphical characterization for non-identifiable causal effects in PAGs building on observations from the completeness proof. The characterization is akin to the ones introduced in (Shpitser & Pearl, 2006; Bareinboim & Pearl, 2012) for a given causal diagram. We start by introducing the following construction, where $\mathbf{R}$ is a root set of $\mathcal{P}$ over $\mathbf{V}$ iff $\mathbf{V} = \texttt{An}(\mathbf{R})_{\mathcal{P}}$ and it is maximal if no subset satisfies the property.

Definition 7 (DC-forest)**.** *Let $\mathcal{P}$ denote any subgraph of a PAG, where $\mathbf{Y}$ is the maximal root set. Then $\mathcal{P}$ is a $\mathbf{Y}$-rooted DC-forest if $\mathcal{P}$ is a dc-component and all nodes have at most one possible child through a directed ($\rightarrow$) or partially directed ($\circ\!\!\rightarrow$) edge.*

For instance, the PAG in Fig. 3c constructs a $\mathbf{Y}$-rooted *dc-forest* where $\mathbf{Y} = \{Y_1, \ldots, Y_4\}$. Using the above, we define the notion of a *$\mathcal{P}$-Hedge* as follows.

Definition 8 ($\mathcal{P}$-Hedge)**.** *Let $\mathbf{X}$, $\mathbf{Y}$ disjoint sets of nodes in PAG $\mathcal{P}$. Let $\mathbf{F}$, $\mathbf{F}'$ be $\mathbf{R}$-rooted DC-forests such that $\mathbf{F} \cap \mathbf{X} \neq \emptyset$, $\mathbf{F}' \cap \mathbf{X} = \emptyset$, $\mathbf{F}' \subset \mathbf{F}$, $\mathbf{R} \subseteq \texttt{An}(\mathbf{Y})_{\mathcal{P}_{\mathbf{V}\setminus\mathbf{X}}}$. Then $\mathbf{F}$ and $\mathbf{F}'$ form a $\mathcal{P}$-hedge for $P_{\mathbf{x}}(\mathbf{Y})$ in $\mathcal{P}$.*

Fig. 3 contains samples of PAGs with non-identifiable effect $P_{\mathbf{x}}(\mathbf{y})$. The simplest example is shown in Fig. 3a in which a proper possibly directed path from X to Y is composed of an invisible edge. Fig. 3c is more complex and contains a $\mathcal{P}$-hedge with $\mathbf{F} = \mathbf{V}$, i.e. all nodes, and $\mathbf{F}' = \{Y_1 \ldots, Y_4\}$.

Theorem 5 (Non-Identifiability Criterion)**.** *Given a PAG $\mathcal{P}$, $P_{\mathbf{x}}(\mathbf{y})$ is non-identifiable in $\mathcal{P}$ if and only if there exist:*

1. *proper possibly directed path from $\mathbf{X}$ to $\mathbf{Y}$ that starts with an invisible edge; or*
2. *dc-forests $\mathbf{F}$, $\mathbf{F}'$ forming a $\mathcal{P}$-hedge for $P_{\mathbf{x}}(\mathbf{y})$.*

6. Conclusion

We tackled the problem of causal identification in a Markov equivalence class represented by a PAG. First, we introduced a novel decomposition strategy for a given causal query. We then used the decomposition to develop an algorithm for causal identification, and showed it to be complete. A significant implication is that whenever an effect is identifiable in every causal diagram compatible with a PAG, the result is the same for all. We also introduced a graphical characterization for non-identifiable causal effects. As the full causal structure is not learnable in many practical settings, we believe these results will be useful for data-intensive sciences where identifying causal effects is important.

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"Causal Identification under Markov Equivalence: Completeness Results" Supplemental Material

In this document, we present the full proofs for the completeness result. We define the following construct from (Zhang, 2008b) as it is used throughout the proof.

Definition 9 (discriminating path)**.** *In a MAG or a PAG, a path $p = \langle X, \ldots, W, V, Y \rangle$ is discriminating for V if:*

1. *p includes at least three edges;*
2. *V is a non-endpoint node on p, and is adjacent to Y on p; and*
3. *X is not adjacent to Y, and every node between X and V is a collider on p and is a parent of Y.*

We start by proving an important property of a *region*. Basically, Lemma 4 states that there is no node in $\mathbf{C}$ which is outside $\mathcal{R}^{\mathbf{C}}_{\mathbf{A}}$ with an invisible edge into the region. It follows that this property holds for the regions in the decomposition queries of $\mathbf{IDP}$-line (10). It also holds for $\mathcal{R}_{\mathbf{B}} \cap \mathcal{R}_{\mathbf{C} \setminus \mathcal{R}_{\mathbf{B}}}$ in the denominator query as the set is an intersection of two regions, so a violation of the property implies the violation for one of the regions.

Lemma 4. *Given a PAG $\mathcal{P}$ over $\mathbf{V}$, $\mathbf{A} \subset \mathbf{C} \subseteq \mathbf{V}$, there doesn't exist a node $Z \in \mathbf{C}$ such that $Z \notin \mathcal{R}^{\mathbf{C}}_{\mathbf{A}} \wedge Y \in \mathcal{R}^{\mathbf{C}}_{\mathbf{A}} \wedge$ invisible $Z{*\!\!\rightarrow} Y$.*

Proof. We prove this property for $\mathbf{C} = \mathbf{V}$, then we generalize it to the case where $\mathbf{C}$ is a subset of $\mathbf{V}$.

Suppose for contradiction that such a node Z does exist. Note that there is an invisible edge from Z into every node in the bucket of Y (Zhang, 2008b, Lemma A.1). By construction of $\mathcal{R}^{\mathbf{C}}_{\mathbf{A}}$, Y must be in a bucket containing at least one node that is in the same pc-component with a node $X \in \mathbf{A}$. Let Y denote one such node, hence we have $X {*\!\!\rightarrow} \leftarrow - \rightarrow W \leftarrow\!\!* Y \leftarrow\!\!* Z$.

The edge between W and Y can't be bi-directed as this contradicts the condition that $Z \notin \mathcal{R}^{\mathbf{C}}_{\mathbf{A}}$. Z and W are adjacent and the edge is into W since the edge between W and Y is either invisible $W \leftarrow Y$ (Zhang, 2006, FCI:R2) or $W \leftarrow\!\circ Y$ (Zhang, 2008b, Lemma A.1). Also, the edge between W and Z is out of Z and visible, else Z is in the same pc-component with X and we violate the condition $Z \notin \mathcal{R}^{\mathbf{C}}_{\mathbf{A}}$. The latter implies that there is a collider path $V_1 \leftarrow - \rightarrow V_{n-1} \leftarrow\!* V_n$ into Z consistent with the graphical condition for visibility of $W \leftarrow Z$ in $\mathcal{P}$. Note here that the edge between Z and Y can't be bi-directed as this would create a discriminating path for Y due to $\langle W, Y, Z, V_1, \ldots, V_n \rangle$ and we get $W \leftrightarrow Y$ or visible $W \leftarrow Y$ both of which lead to contradictions (Zhang, 2008b, FCI:R4).

Let V_i be the first node along $\langle V_1, \ldots, V_n \rangle$ that is not adjacent to Y. If V_i is not defined, then Y is adjacent to all the nodes along $\langle V_1, \ldots, V_n \rangle$. We show, by induction, that all the edges between Y and V_j, $1 \leq j < i$, are into Y. In the base case, the edge between Y and V_1 is into Y by (Zhang, 2006, FCI:R2) if $Y \leftarrow Z$ and (Zhang, 2008b, Lemma A.1) if $Y \leftarrow\!\circ Z$. In the induction step, suppose the property holds for up to V_j, we prove it for V_{j+1}. If the edge between Y and V_j is bi-directed, then we have a discriminating path for Y due to $\langle W, Y, V_j, \ldots, V_n \rangle$ and we get $W \leftrightarrow Y$ or visible $W \leftarrow Y$ both leading to a contradiction. It follows that we have $Y \leftarrow\!* V_{j+1}$, else we violate an invariant in $\mathcal{P}$ (Maathuis & Colombo, 2015, Lemma 7.5) due to $V_j \leftarrow\!* V_{j+1}$ and a possibly directed path $\langle V_j, Y, V_{j+1} \rangle$.

If V_i is not defined, i.e. V_n is adjacent to Y, then we have $W \leftrightarrow Y$ or visible $W \leftarrow Y$ (Zhang, 2006, FCI:R0-R1) since W and V_n are not adjacent. Both options lead to a contradiction and hence are not possible. Alternatively, V_i is defined and we have $Y \leftarrow\!* V_j$ for all $1 \leq j < i$ as shown earlier. If any of the edges is $Y \leftrightarrow V_j$, then we have a discriminating path for Y due to $\langle W, Y, V_j, \ldots, V_n \rangle$ and we get $W \leftrightarrow Y$ or visible $W \leftarrow Y$ both leading to a contradiction. Finally, we have $Y \leftarrow\!* Z \leftarrow - \rightarrow V_{i-1} \leftarrow\!* V_i$ and all the edges $Y \leftarrow\!* V_j$ for $1 \leq j < j$ are not bi-directed. It is easy to show, by induction, that the edge is out of V_j for $1 \leq j < i$ ($Y \leftarrow V_j$) (Zhang, 2006, FCI:R1,R4). Hence, we have a discriminating path for Z due to $\langle Y, Z, \ldots, V_i \rangle$ and we get $Y \leftrightarrow Z$ or visible $Y \leftarrow Z$ both leading to a contradiction.

Consequently, the initial assumption that Z exists is not possible, which proves the claim for $\mathbf{C} = \mathbf{V}$. For $\mathbf{C} \subset \mathbf{V}$, the result follows by the above argument and (Jaber et al., 2018a, Lemmas 4, 5). $\square$

Proof of Lemma 1. Suppose for contradiction that the claim doesn't hold – there is no node $X \in \mathbf{T}$ in a failing instance of `Identify`$(\mathbf{C}, \mathbf{T}, Q)$. Then, $\mathbf{T}$ doesn't include any node from $\mathbf{V} \setminus \mathbf{D}$ since all the possible children of $\mathbf{V} \setminus \{\mathbf{D} \cup \mathbf{X}\}$ are in $\mathbf{V} \setminus \mathbf{D}$ by construction of set $\mathbf{D}$ in **IDP**. So, if all $\mathbf{X}$ are gone, then the buckets in $\mathbf{V} \setminus \{\mathbf{D} \cup \mathbf{X}\}$ will satisfy the intervention criterion (**IDP**; Line 6) by reverse order of any PTO. Therefore, we have $\mathbf{T} \subset \mathbf{D}$ and the split of $\mathbf{T}$ into $\mathbf{C}$ and $\mathbf{T} \setminus \mathbf{C}$ is due to decomposition (Corol. 1).

Note here that every bucket in $\mathbf{T}$ is either in $\mathbf{C}$ or in $\mathbf{T} \setminus \mathbf{C}$ by construction of *Region* (Def. 6).

Let $\mathbf{B}_1 < \cdots < \mathbf{B}_m$ be a partial order over $\mathcal{P}_{\mathbf{T}\setminus\mathbf{C}}$. **IDP** can't intervene on $\mathbf{B}_m$ and the condition at line 6 fails, hence some node $B \in \mathbf{B}_m$ is in the same pc-component with a possible child $B{*}\!\!\rightarrow Z$. Node $Z \in \mathbf{C}$ by the choice of $\mathbf{B}_m$, then $B{*}\!\!\rightarrow Z$ must be directed and visible (Lemma 4). Hence, B and Z are in the same pc-component through a collider path consistent with Def. 4. Let W be the node closest Z along the latter path. Now, if $W \in \mathbf{C}$, then all the nodes along the collider path up to and including B will be in $\mathbf{C}$ else we violate Lemma 4. The latter leads to a contradiction since $B \in \mathbf{T} \setminus \mathbf{C}$, so $W \notin \mathbf{C}$. Similarly, the invisible edge $Z{*}\!\!\rightarrow W$ can't be bi-directed. If we have $Z \circ\!\!\rightarrow W$, then B and W are adjacent and the edge is into W and not into B by (Zhang, 2008b, Lemma A.1). Also, if the edge is out of Z, then B and W are adjacent since $Z \rightarrow W$ is invisible and we have $B \rightarrow W$ due to (Zhang, 2008b, FCI:R2,R8). Either case leads to a contradiction since $W \notin \mathbf{C}$ is a possible child of $B \in \mathbf{B}_m$ violating the choice of $\mathbf{B}_m$ in the partial order. Hence the initial assumption fails and this concludes the proof. $\square$

Proof of Lemma 2. The construction follows that in (Zhang, 2008b, Theorem 2), and every bucket can be oriented independently. Also, by **IDP** steps and Condition 1, $\mathcal{P}_{\mathbf{T}\setminus\mathbf{C}}$ and $\mathcal{P}_{\mathbf{C}}$ are induced subgraphs over subsets of the buckets in $\mathcal{P}$. The failing condition of $\texttt{Identify}(\mathbf{C}, \mathbf{T}, Q)$ (**IDP** - line 6) ensures that every bucket in $\mathcal{P}_{\mathbf{T}\setminus\mathbf{C}}$ contains at least one node that satisfies the condition of node B in Step 3 of the current theorem. Orienting all edges in the corresponding bucket out of B is possible by (Maathuis & Colombo, 2015, Lemma 7.6). Last, by failing the condition at line 9 of **IDP**, we have $\mathcal{R}_{\mathbf{B}_m} = \mathbf{C}$. But $\mathbf{B}_m$ only has arrowheads incident on it in $\mathcal{P}_{\mathbf{C}}$, then $\mathcal{R}_{B^*} = \mathbf{C}$ by (Jaber et al., 2018a, Lem. 5). It follows that B^* is in the same pc-component with at least one node in every bucket in $\mathcal{P}_{\mathbf{C}}$. Hence, the condition in Step 4b is satisfied in every bucket in $\mathcal{P}_{\mathbf{C}}$. This concludes the proof. $\square$

Proof of Lemma 3. This follows from Lemmas 5 and 8. $\square$

Lemma 5. *Let $\mathcal{M}$ be the MAG constructed from PAG $\mathcal{P}$ according to Lemma 2. Also, let $\mathbf{T}'$ and $\mathbf{C}'$ denote the set of nodes that are chosen in the orientation of the buckets of $\mathcal{P}_{\mathbf{T}}$ and $\mathcal{P}_{\mathbf{C}}$, respectively. Then, the induced subgraph over $\mathbf{T}'$, denoted $\mathcal{M}^{\mathbf{C}'}_{\mathbf{T}'}$, maintains the following properties:*

1. *every node in $\mathbf{T}' \setminus \mathbf{C}'$ has a child such that the edge in between is invisible or the two nodes share a bi-directed path;*
2. *$\mathcal{R}^{\mathbf{C}'}_N = \mathbf{C}'$ for every $N \in \mathbf{C}'$; and*
3. *$\mathbf{T}' \setminus \mathbf{C}'$ contains at least one intervention $\mathcal{X} \in \mathbf{X}$.*
4. *$\mathbf{C}' \subseteq An(\mathbf{Y})_{\mathcal{M}_{\mathbf{V}\setminus\mathbf{X}}}$.*

Proof. Let B be the node chosen for orientation in any bucket in $\mathcal{P}_{\mathbf{T}\setminus\mathbf{C}}$. By the condition in Lemma 2- Step 3, B is in the same pc-component with a possible child and both nodes are in different buckets. If B has a possible child C where $B{*}\!\!\rightarrow C$ is invisible, then every node in the corresponding bucket of C is a possible child of B by (Jaber et al., 2018a, Lem. 5) and all those edges are not visible since $B{*}\!\!\rightarrow C$ is not visible. It follows that B is in the same pc-component with its possible child C^*, the chosen node for orientation in the corresponding bucket of C. The edge between B and C^* remains invisible in $\mathcal{M}_{\mathbf{T}'}$ by (Perković et al., 2018, Lemma B.1). Otherwise, we have a visible edge $B \rightarrow C$ and B and C share a collider path in $\mathcal{P}_{\mathbf{T}}$ consistent with Def. 4, i.e. $B \leftrightarrow T_1 \leftarrow - - \rightarrow T_n \leftarrow\!\!{*}C$. Note that B and T_n are adjacent if $T_n \leftarrow\!\!{*}C$ is not bi-directed since the latter is invisible, the edge is into T_n (Zhang, 2008b, FCI:R2), and the edge is not into B by (Maathuis & Colombo, 2015, Lemma 7.5). It follows that B is in the same dc-component with a possible child in $\mathcal{P}_{\mathbf{T}}$. By (Zhang, 2006, Lemma 3.3.2), there is a bi-directed edge between every pair of nodes from two buckets if there is one bi-directed edge between them. Hence, B is in the same dc-component with a child in $\mathcal{M}_{\mathbf{T}'}$.This concludes the proof of Property 1

For Property 2, note that N is in the same pc-component with B^* in $\mathcal{P}_{\mathbf{C}}$ by construction in Lemma 2 - Step 4b. Hence, N is in the same pc-component with B^* in $\mathcal{M}_{\mathbf{C}'}$ by (Perković et al., 2018, Lemma B.1). But, B^* only has arrowheads incident on it in $\mathcal{M}_{\mathbf{C}'}$ since $B^* \in \mathbf{B}_m$ in the PTO over $\mathcal{P}_{\mathbf{C}}$, so the collider path between N and B^* is into B^*. Also, every node in $\mathbf{C}'$ is in the same pc-component with B^* in $\mathcal{P}_{\mathbf{C}}$ by construction, and consequently in $\mathcal{M}_{\mathbf{C}'}$. Hence, $\mathcal{R}^{\mathbf{C}'}_N = \mathbf{C}'$.

Next, we prove Property 3. By Lemma 1, $\mathcal{P}_{\mathbf{T}}$ contains at least one bucket $\mathbf{B}$ such that $\mathbf{B}$ contains an intervention $X \in \mathbf{X}$, and this bucket is in $\mathbf{V} \setminus \mathbf{D}$ by Conditon 1. Hence, at least one node from $\mathbf{B}$ is in $\mathcal{M}_{\mathbf{T}'}$. Let $B_1 < \cdots < B_m$ be a topological order over the induced subgraph of $\mathcal{M}$ over $\mathbf{T}' \cap (\mathbf{V} \setminus \mathbf{D})$. Node B_m is in $\mathbf{T}' \setminus \mathbf{C}'$ by construction and it is in the same pc-component with a child by Step 1. The child is in $\mathbf{D}$ due to the topological order and the choice of B_m, hence $B_m \in \mathbf{X}$.

Finally, note that $\mathbf{C}' \subseteq \mathbf{D} = \texttt{An}(\mathbf{Y})_{\mathcal{P}_{\mathbf{V}\setminus\mathbf{X}}}$ by construction in line (1) of **IDP**, and no pair of nodes in $\mathbf{C}'$ belong to the same bucket in $\mathcal{P}$ by the construction of $\mathcal{M}$ in Lem. 2. Then, for any $C \in \mathbf{C}'$, there is an uncovered potentially directed path from C to $\mathbf{Y}$ denoted p (Zhang, 2008b, Lemma B.1). Since we orient all edges out of C in $\mathcal{M}$ (Lem. 2-step 4), then the path corresponding to p in $\mathcal{M}$ is a directed path out of C. Therefore, $\mathbf{C}' \subseteq \texttt{An}(\mathbf{Y})_{\mathcal{M}_{\mathbf{V}\setminus\mathbf{X}}}$. $\square$

Lemma 6. *In $\mathcal{M}_{\mathbf{A}}$, where $\mathcal{M}$ is a MAG over $\mathbf{V}$ and $\mathbf{A} \subseteq \mathbf{V}$, the following property holds:*

For any three vertices A, B, C, if $A {*\!\!\rightarrow} B \rightarrow C$ and both edges are invisible, then we have $A {*\!\!\rightarrow} C$ and the edge is invisible.

Proof. First, we prove this property when $\mathbf{A} = \mathbf{V}$. Edge $B \rightarrow C$ is invisible, then A and C must be adjacent. Also, the edge must be into C, otherwise the MAG is not ancestral. If the edge between A and C is bi-directed or directed and invisible, then we are done. Otherwise, suppose for contradiction that $A \rightarrow C$ is visible. The edge between A and B is out of A, otherwise $B \rightarrow C$ is visible due to the visibility of $A \rightarrow C$ and we reach a contradiction. Consider the collider path between some node F and A that is into A and is indicating the visibility of $A \rightarrow C$ (Zhang, 2008a, Def. 8). If any of the nodes along the collider path including F is adjacent to B with a bi-directed edge, then $B \rightarrow C$ is visible which is a contradiction. Alternately, it is easy to show by induction that every node along the collider path is a parent of B including F, otherwise $A \rightarrow B$ is visible and we have a contradiction. Now, we have $F \rightarrow B \rightarrow C$ and F and C are not adjacent. Hence, $B \rightarrow C$ is visible and we reach a contradiction. So, the assumption that $A \rightarrow C$ is visible doesn't hold which concludes the proof. Since the property holds for a full MAG, then it trivially holds in a corresponding induced subgraph. $\square$

Lemma 7. *Let $\mathcal{M}^{\mathbf{C}}_{\mathbf{T}}$ denote the induced sub-MAG considered in Lemma 5, where $\mathbf{T} = \mathbf{T}'$ and $\mathbf{C} = \mathbf{C}'$, and let N be any node in $\mathbf{T}$. Then, the pc-component of N in $\mathcal{M}^{\mathbf{C}}_{\mathbf{T}}$ is $\mathbf{T}$.*

Proof. Let C^* be a node in $\mathcal{M}_{\mathbf{C}}$ with only arrowheads incident on it. Such a node exists since MAGs don't have cycles. We show that every node in $\mathcal{M}^{\mathbf{C}}_{\mathbf{T}}$ is in the same pc-component with C^* through a collider path into C^*. This implies the claim of the lemma.

Let W be any node in $\mathbf{T}$ other than C^*. If $W \in \mathbf{C}$, then W is in the same pc-component with C^* by the second condition of Lemma 5. Otherwise, W shares a directed path with some node $Z \in \mathbf{C}$ along which every pair of consecutive nodes either share a directed invisible edge or a bi-directed path (condition 1 of Lem. 5). In what follows, we show by induction on the length of the directed path that W is in the same pc-component with every node along the path including Z and the collider path is into the descendant node. The base case is trivial as either the edge between W and its child is invisible or both nodes share a bi-directed path. In the induction step, suppose that the assumption holds until node T_m along the path where $W = T_1$ and we prove it for T_{m+1}. By the induction step, we have that W and T_m are in the same pc-component and the collider path is into T_m. If any of the nodes along the collider path including W is connected to T_{m+1} with a bi-directed edge, or T_m and T_{m+1} share a bi-directed path, then we are done. Alternatively, note that we have $T_m \rightarrow T_{m+1}$ and the edge is invisible. Hence, every node along the collider path including W is a parent of T_{m+1}, else $T_m \rightarrow T_{m+1}$ is visible and we have a contradiction. If $W \rightarrow T_{m+1}$ is invisible then we are done. Otherwise, assume that $W \rightarrow T_{m+1}$ is visible. Then, we have $W {*\!\!\rightarrow} A \rightarrow T_{m+1}$ and $W \rightarrow T_{m+1}$ with the latter visible and A denoting the first node along the collider path after W. If $A \rightarrow T_{m+1}$ is visible, then $T_m \rightarrow T_{m+1}$ is visible which is not true. Also, $W {*\!\!\rightarrow} A$ is invisible by Def. 4. This leads to a contradiction (Lemma 6) and $W \rightarrow T_{m+1}$ can't be visible. This concludes the induction proof.

By the above induction, we have that W is in the same pc-component with Z and the collider path is into Z. If Z is in the dc-component of C^*, then we are done. Otherwise, Z is in the pc-component with C^* and the collider path is into C^*, i.e. the first edge along the collider path is out of Z, due to the choice of C^*. An argument by induction similar to the one earlier proves that W is in the pc-component of the child of Z and the collider path is into the child. Hence, W is in the pc-component of C^* and the path is into C^*. This concludes the proof. $\square$

Lemma 8. *Let $\mathcal{M}^{\mathbf{C}}_{\mathbf{T}}$ denote the induced sub-MAG considered in Lemma 5. Then, there is a corresponding induced subgraph $\mathcal{M}^{\mathbf{C}'}_{\mathbf{T}'}$, where $\mathbf{C}' \subseteq \mathbf{C}$ and $\mathbf{T}' \subseteq \mathbf{T}$, that maintains the following properties:*

1. *every node in $\mathbf{T}' \setminus \mathbf{C}'$ is in the same dc-component with a child;*
2. *$\mathbf{C}'$ is a single dc-component in $\mathcal{M}_{\mathbf{C}'}$; and*
3. *$\mathbf{T}' \setminus \mathbf{C}'$ contains at least one intervention $\mathcal{X} \in \mathbf{X}$.*
4. *$\mathbf{C}' \subseteq \mathit{An}(\mathbf{Y})_{\mathcal{M}_{\mathbf{V} \setminus \mathbf{X}}}$.*

Proof. We start with the second property of the lemma. Let C^* be a node in $\mathcal{M}_{\mathbf{C}}$ with only arrowheads incident on it. Such a node exists since MAGs don't have cycles. By condition 2 of Lemma 5, every node in $\mathcal{M}_{\mathbf{C}}$ is in the pc-component of C^*. In $\mathcal{M}_{\mathbf{C}}$, let J be any node that is in the pc-component of C^* but not in its dc-component and let K be the child of J along the collider path. If any other node in $\mathcal{M}_{\mathbf{C}}$ is in the pc-component of C^* through J, then J is in the dc-component of C^* and we have a contradiction. Hence, dropping J doesn't affect condition 2 of Lemma 5. It remains to show that dropping J doesn't affect condition 1 of Lemma 5. Suppose for contradiction that some node $Z \in \mathbf{T} \setminus \mathbf{C}$ is not in the same pc-component with a child in the induced subgraph of $\mathcal{M}^{\mathbf{C}}_{\mathbf{T}}$ over $\mathbf{T} \setminus J$.

First case is that J is a child of Z through an invisible edge. We have $Z \to J \to K$ and both edges are invisible. Hence, we have invisible edge $Z \to K$ by Lemma 6. This violates the choice of Z since it is in the pc-component with a child without J and we have a contradiction.

The other option is that Z is in the dc-component with a child W through a bi-directed path that goes through J. Consider the collider path between Z and J and recall that $J \to K$ is invisible. If none of the nodes along the collider path is adjacent to K with a bi-directed edge, then each node along the path including Z is a parent of K, otherwise $J \to K$ would be visible. Also, $Z \to K$ is invisible since $J \to K$ is invisible, and we go back to the first case which leads to contradictions. Hence, some node along the collider path between Z and J is adjacent to K with a bi-directed edge and K is in the dc-component of Z. Now, consider the collider path between W and J. If some node along the path is adjacent to K with a bi-directed edge, then Z is in the dc-component with its child W without going through J. This contradicts our assumption that Z fails condition 1 of Lemma 5. Otherwise, every node along the collider path including W is a parent of K and $W \to K$ is invisible since $J \to K$ is invisible. We have $Z \to K$ due to $Z \to W$ and invisible $W \to K$. Hence, Z is in the dc-component with its child K and the path doesn't go through J which is a contradiction.

This concludes the proof that dropping J doesn't affect condition 1 of Lemma 5. This argument can be applied recursively until all the nodes in $\mathbf{C}' \subseteq \mathbf{C}$ are in the dc-component of C^* in $\mathcal{M}_{\mathbf{C}'}$.

Next, we prove the first property. Note that $\mathcal{X}$ in condition 3 of Lem. 5, by construction, doesn't have any invisible edges out of it in $\mathcal{M}_{\mathbf{T}}$, else we violate Cond. 1. Also, every node in $\mathcal{M}_{\mathbf{T}}$ is in the pc-component $\mathcal{X}$ by Lem. 7. Hence, every node in $\mathcal{M}_{\mathbf{T}}$ is in the pc-component of $\mathcal{X}$ through a collider path into $\mathcal{X}$. With this property, let J be any node in $\mathbf{T} \setminus \mathbf{C}$ that is in the pc-component of $\mathcal{X}$ but not its dc-component and let K be the child of J along the collider path. Suppose for contradiction that some node $Z \in \mathbf{T} \setminus \mathbf{C}$ violates condition 1 of Lem. 5 in the induced subgraph of $\mathcal{M}^{\mathbf{C}}_{\mathbf{T}}$ over $\mathbf{T} \setminus J$. This case was handled earlier and we reach a contradiction. Hence, we can drop J from the induced subgraph without violating condition 1 of Lem. 5.

Moreover, if any other node is in the pc-component of $\mathcal{X}$ through J, then J would be in the dc-component of $\mathcal{X}$ and we have a contradiction. Hence, dropping J from the induced subgraph doesn't violate Lemma 7. It follows that we can apply the previous argument recursively and drop nodes until every node in $\mathbf{T}' \setminus \mathbf{C}'$ is in the dc-component of $\mathcal{X}$. Hence, every node in $\mathbf{T}' \setminus \mathbf{C}'$ is in the dc-component with a child through $\mathcal{X}$. The third property is trivially satisfied by our choice of $\mathcal{X}$ above. Also, the fourth property is trivially satisfied since $\mathbf{C}' \subseteq \mathbf{C}$ in $\mathcal{M}$. $\square$

Proof of Theorem 5. **(if)** Case 1 implies non-identifiability by Th. 3. As for case 2, let $\mathbf{R}' \subseteq \mathbf{R}$ such that no pair of nodes in $\mathbf{R}'$ belong to the same bucket in the full PAG $\mathcal{P}$. The latter is needed to ensure later that $\mathbf{R}' \subseteq \mathtt{An}(\mathbf{Y})_{\mathcal{M}_{\mathbf{V} \setminus \mathbf{X}}}$ in some MAG $\mathcal{M}$ in the equivalence class of $\mathcal{P}$. Let $\mathcal{F}$ and $\mathcal{F}'$ denote the corresponding induced subgraphs of $\mathcal{P}$ over the nodes in $\mathbf{F}$ and $\mathbf{F}'$, respectively, excluding $\mathbf{R} \setminus \mathbf{R}'$. By (Jaber et al., 2018a, Lem. 5), both $\mathcal{F}$ and $\mathcal{F}'$ preserve the properties of a $\mathcal{P}$-Hedge except for the one where each node has at most single child. Let $\mathcal{M}$ denote the MAG constructed from $\mathcal{P}$ by (1) orienting partially directed edges $A \circ\!\!\to B$ out of A and (2) orient each bucket into a DAG with no unshielded colliders. $\mathcal{M}$ is in the equivalence class of $\mathcal{P}$ since the construction is a special case of Lemma 2. Next, we construct a causal diagram $\mathcal{D}$ from $\mathcal{M}$ by keeping directed and bi-directed edges intact. The diagram is trivially in the equivalence class of $\mathcal{M}$. Finally, the induced subgraphs of $\mathcal{D}$ corresponding to $\mathcal{F}$ and $\mathcal{F}'$ preserve the properties of a hedge ((Shpitser & Pearl, 2006, Def. 7)) except for each node having at most single child. We can remove directed edges to establish the latter property, hence the effect is not identifiable in $\mathcal{D}$ and consequently in $\mathcal{P}$.

(only if) If the effect is non-identifiable, then **IDP** fails. The first case of the current theorem follows if Cond. 1 in the completeness proof doesn't hold. Otherwise, we can construct a MAG in the equivalence class with the properties in Lem. 3. But the MAG construction procedure in Lemma 2 does not introduce any bi-directed edges other than the ones already present in the PAG. Also, every node in $\mathcal{M}_{\mathbf{T}'}$ (Lem. 3) is from a bucket in $\mathcal{P}$ so all the directed edges between $\mathbf{T}'$ correspond to directed or partially directed edges in $\mathcal{P}$. Hence, $\mathbf{T}', \mathbf{C}'$ in Lem. 3 construct a $\mathcal{P}$-hedge for $P_{\mathbf{x}}(\mathbf{y})$ in $\mathcal{P}$ after dropping some (partially) directed edges. $\square$

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